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Joint Probability Density Function of Selected Order Statistics and the Sum of the Remaining Random Variables

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A set of N independent, identically distributed random variables $\{x_n\}$, with common probability density function p(x), are ordered into a new set of dependent random variables $\{x'_n\}$, each with a different probability density function. From this latter set, the n_1–th largest random variable through the n_{M-1}-th largest random variable are selected. Then, the sum of the remaining N+1-M random variables is computed, giving a total of M dependent random variables.

The joint probability density function of these M random variables is derived in a form involving a single Bromwich contour integral in the moment-generating function domain. The integral is most easily numerically evaluated by locating (approximately) the real saddlepoint of the integrand and passing the contour through this point. Very high accuracy in the probability density function evaluation is available by using numerical integration instead of a saddlepoint approximation.

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LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS

$c(u,\lambda)$	Generalized cumulative distribution, equation (3)
$c_h(u,\lambda)$	Auxiliary function, equation (46)
C	Bromwich contour in λ-plane, equation (14)
CDF	Cumulative distribution function
CGF	Cumulant generating function
D	Constant, equation (43)
$e(u,\lambda)$	Generalized exceedance distribution, equation (4)
$e_h(u,\lambda)$	Auxiliary function, equations (46) and (50)
E	Expectation, equation (7)
EDF	Exceedance distribution function
F	Number of events, equation (39)
FO	First order
g	Joint PDF of ordered random variables, equation (2)
$h(x)$	Nonlinear transformation, equation (46)
IID	Independent, identically distributed
I_n	n-th integral, equations (8) through (12)
$I(\lambda)$	Integrand function, equation (44)
M	Number of random variables of interest, equation (1)
MGF	Moment-generating function
n_m	n_m-th largest random variable, m=1:M-1, equation (1)
N	Number of original random variables $\{\mathbf{x}_n\}$
N(0,1)	Normalized Gaussian random variable, equation (47)
$(N\|M)$	Binomial coefficient N!/[M! (N-M)!], equation (28)
$p(x)$	Probability density function of $\mathbf{x}_n$
$p_c(x)$	Conditional probability density, equation (20)

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

$p_e(x)$	Conditional probability density, equation (30)
$p_{\mathbf{r}}$	Probability density function of ratios, equation (53)
$p_{\mathbf{z}}(z_1,z_2)$	Probability density of random vector **z**, equation (15)
PDF	Probability density function
$q(u)$	Delta function, equation (14)
$\mathbf{r}_m$	m-th ratio of random variables, equation (51)
$\mathrm{Re}(\lambda)$	Real part of λ
RV	Random variable or random vector, equation (1)
SO	Second order
TO	Third order
u_n	n-th argument of probability densities, equation (2)
U	Unit step function, equation (35)
w	Auxiliary probability function, appendix A
$\mathbf{x}_n$	n-th original random variable, n=1:N, equation (1)
$\mathbf{x}'_n$	n-th ordered random variable, equation (1)
y	Variable of integration, equation (45)
$\mathbf{z}$	M-dimensional random vector, equation (1)
$\mathbf{z}_m$	m-th random variable, equation (1)
z	Field point (nonrandom vector), equation (15)
z_m	m-th component of field point z, equation (15)
boldface	Random variable or random vector

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

λ	Argument of c function, equation (3)
λ_m	m-th argument of moment function, equation (7)
λ_s	Saddlepoint location, equation (45)
Φ	Cumulative Gaussian distribution, appendix A
$\mu(u_a, u_b, \lambda)$	Interval moment-generating function, equation (40)
$\mu(\lambda)$	Moment-generating function of $\mathbf{x}_n$, equations (3), (5)
$\mu_z(\lambda)$	Moment-generating function of $\mathbf{z}$, equation (7)

JOINT PROBABILITY DENSITY FUNCTION OF SELECTED ORDER STATISTICS AND THE SUM OF THE REMAINING RANDOM VARIABLES

INTRODUCTION

Detection and location of weak signals in random noise is frequently accomplished by the ordering of the random variables (RVs) in a measured data set, followed by an investigation of the locations and statistics of several of the largest RVs under consideration. Also of interest are the remaining smaller RVs in the data set, which can be used to estimate the background noise level and to form a basis for normalization, thereby realizing a constant false alarm processor.

In this study, the original data set $\{\mathbf{x}_n\}$ is composed of N independent, identically distributed (IID) RVs with a common arbitrary probability density function (PDF) p(x). This data set is ordered into the modified data set $\{\mathbf{x}'_n\}$, n=1:N, of dependent RVs, where $\mathbf{x}'_1 \geq \mathbf{x}'_2 \geq \cdots \geq \mathbf{x}'_N$. Then, M-1 RVs are selected from this latter data set, namely, the n_1-th largest RV, the n_2-th largest RV, ..., and the n_{M-1}-th largest RV ($M \geq 2$), where, without loss of generality, $1 \leq n_1 < n_2 < \cdots < n_{M-1} \leq N$. Finally, the sum of the remaining RVs in the ordered data set is formed for a total of M dependent RVs. The joint M-th order PDF of these M dependent RVs is the quantity of interest.

For convenience of notation, the M-dimensional random vector (RV) $\mathbf{z} = [\mathbf{z}_1 \cdots \mathbf{z}_M]$ is defined as

$$\mathbf{z}_1 = \mathbf{x}'_{n_1} , \ldots , \mathbf{z}_{M-1} = \mathbf{x}'_{n_{M-1}} , \quad \mathbf{z}_M = {\sum_{n=1}^{N}}' \mathbf{x}'_n , \tag{1}$$

where the tic mark on the sum denotes $n \neq n_1, n_2, \ldots, n_{M-1}$. Thus, the first M-1 components of RV $\mathbf{z}$ satisfy the inequalities $\mathbf{z}_1 \geq \mathbf{z}_2 \geq \cdots \geq \mathbf{z}_{M-1}$. The joint PDF of M-dimensional RV $\mathbf{z}$ is of interest here.

To determine this joint PDF of RV $\mathbf{z}$, a series of simpler problems will be solved first, and the results will be interpreted in terms of conditional PDFs. From these simpler results, the general form of the M-dimensional joint PDF of RV $\mathbf{z}$ in equation (1) can be deduced. The end result is a single one-dimensional contour integral for the joint PDF of $\mathbf{z}$, which can easily and accurately be numerically evaluated by moving the contour of integration to pass through the saddlepoint of the integrand. As a backup, an alternative approach involving the joint M-dimensional moment-generating function (MGF) of RV $\mathbf{z}$ will also be derived; however, it is not as useful as the single contour integral indicated above.

Since the original RVs $\{\mathbf{x}_n\}$ are IID with common PDF $p(x)$, the N-dimensional joint PDF of the ordered RVs $\{\mathbf{x}'_n\}$, n=1:N, is simply

$$g(u_1, \ldots, u_N) = N!\ p(u_1) \cdots p(u_N) \quad \text{for } u_1 > u_2 > \cdots > u_N \tag{2}$$

and zero elsewhere. The requirement that $u_n > u_{n+1}$ for n=1:N-1 represents the statistical dependence amongst the ordered RVs.

For later use, it is convenient to define two auxiliary functions. First, using the common PDF p(x) of RV $\mathbf{x}_n$, function

$$c(u,\lambda) = \int_{-\infty}^{u} dx\ p(x)\ \exp(\lambda x) \qquad (3)$$

is defined, which is a mixture of a cumulative distribution function (CDF) and an MGF. That is, c(u,0) is the first-order (FO) CDF of RV $\mathbf{x}_n$, while $c(+\infty,\lambda)$ is the FO MGF $\mu(\lambda)$ of RV $\mathbf{x}_n$, n=1:N. Variable u is real, while λ can be complex.

Also defined is the auxiliary function

$$e(u,\lambda) = \int_{u}^{+\infty} dx\ p(x)\ \exp(\lambda x)\ , \qquad (4)$$

which is a mixture of an exceedance distribution function (EDF) and an MGF. That is, e(u,0) is the FO EDF of RV $\mathbf{x}_n$, and $e(-\infty,\lambda)$ is the FO MGF $\mu(\lambda)$ of RV $\mathbf{x}_n$, n=1:N. The two auxiliary functions are interrelated according to

$$c(u,\lambda) = e(-\infty,\lambda) - e(u,\lambda) = \mu(\lambda) - e(u,\lambda)\ , \qquad (5)$$

$$e(u,\lambda) = c(+\infty,\lambda) - c(u,\lambda) = \mu(\lambda) - c(u,\lambda)\ , \qquad (6)$$

where $\mu(\lambda)$ is the common MGF corresponding to PDF p(x). Several useful examples of the $c(u,\lambda)$ and $e(u,\lambda)$ functions are listed in appendix A.

SOME SPECIAL CASES

JOINT PDF OF THE LARGEST RV AND SUM OF REMAINING RVs

For this special case, M = 2. The second-order (SO) MGF of RV $\mathbf{z} = [\mathbf{z}_1 \ \mathbf{z}_2]$ is given by the expectation

$$\mu_{\mathbf{z}}(\lambda_1,\lambda_2) = \mathrm{E}\ \exp(\lambda_1 \mathbf{z}_1 + \lambda_2 \mathbf{z}_2) = \mathrm{E}\ \exp\left(\lambda_1 \mathbf{x}_1' + \lambda_2 \sum_{n=2}^{N} \mathbf{x}_n'\right)$$

$$= \int \cdots \int du_1 \cdots du_N \ g(u_1,\ldots,u_N) \ \exp\left(\lambda_1 u_1 + \lambda_2 \sum_{n=2}^{N} u_n\right)$$

$$= N! \int_{-\infty}^{\infty} du_1 \ p(u_1) \ \exp(\lambda_1 u_1) \int_{-\infty}^{u_1} du_2 \ p(u_2) \ \exp(\lambda_2 u_2) \ \cdots$$

$$\times \int_{-\infty}^{u_{N-1}} du_N \ p(u_N) \ \exp(\lambda_2 u_N) \ . \tag{7}$$

Denote the integral on variable u_n by I_n. Then, by reference to equation (3),

$$I_N = c(u_{N-1},\lambda_2) \ . \tag{8}$$

Next, there follows

$$I_{N-1} = \int_{-\infty}^{u_{N-2}} du_{N-1} \ p(u_{N-1}) \ \exp(\lambda_2 \ u_{N-1}) \ c(u_{N-1},\lambda_2)$$

$$= \int_{-\infty}^{u_{N-2}} du_{N-1} \ c'(u_{N-1},\lambda_2) \ c(u_{N-1},\lambda_2) = \frac{1}{2} \left[c(u_{N-2},\lambda_2)\right]^2 \ . \tag{9}$$

Continuing in this manner leads to

$$I_{N-2} = \frac{1}{3!}\left[c(u_{N-3},\lambda_2)\right]^3 \ , \ldots , \tag{10}$$

$$I_2 = \frac{1}{(N-1)!}\left[c(u_1,\lambda_2)\right]^{N-1} \ , \tag{11}$$

or, generally, to

$$I_n = \frac{1}{(N+1-n)!}\left[c(u_{n-1},\lambda_2)\right]^{N+1-n} \quad \text{for } n=2:N \ . \tag{12}$$

Finally, the SO MGF of RV $\mathbf{z}$ is

$$\mu_{\mathbf{z}}(\lambda_1,\lambda_2) = N \int_{-\infty}^{\infty} du_1 \ p(u_1) \ \exp(\lambda_1 \ u_1) \left[c(u_1,\lambda_2)\right]^{N-1} \ . \tag{13}$$

This single integral must be numerically evaluated to determine the SO MGF $\mu_{\mathbf{z}}(\lambda_1,\lambda_2)$.

For PDF $q(u) = \delta(u - a)$, the MGF is $\exp(\lambda \ a)$, and the inverse Laplace transform is given by the Bromwich contour integral on C:

$$q(u) = \frac{1}{i2\pi} \int_C d\lambda \ \exp(-\lambda u) \ \exp(\lambda a) = \delta(u - a) \ . \tag{14}$$

This is an important reference case and will be used frequently.

The SO PDF of RV $\mathbf{z}$ at arbitrary field point $z = [z_1 \ z_2]$ is

$$p_{\mathbf{z}}(z_1,z_2) = \frac{1}{(i2\pi)^2} \iint_C d\lambda_1 \ d\lambda_2 \ \exp(-\lambda_1 z_1 - \lambda_2 z_2) \ \mu_{\mathbf{z}}(\lambda_1,\lambda_2)$$

$$= \frac{N}{(i2\pi)^2} \iint_C d\lambda_1 \, d\lambda_2 \, \exp(-\lambda_1 z_1 - \lambda_2 z_2) \int_{-\infty}^{\infty} du_1 \, p(u_1) \, \exp(\lambda_1 u_1)$$

$$\times \left[c(u_1, \lambda_2)\right]^{N-1} = N \, p(z_1) \, \frac{1}{i2\pi} \int_C d\lambda \, \exp(-\lambda z_2) \, \left[c(z_1, \lambda)\right]^{N-1} \, , \qquad (15)$$

where use has been made of equation (14).

Since the RVs must always satisfy $\mathbf{z}_2 < (N-1)\, \mathbf{z}_1$, the field point $z = [z_1 \; z_2]$ should be chosen so that field components $z_2 < (N-1)\, z_1$; otherwise, the integral result in equation (15) will yield zero. This result may be deduced directly from the integral for SO PDF $p_{\mathbf{z}}(z_1, z_2)$ in equation (15) as follows. Function $c(u, \lambda)$ in equation (3) is analytic in λ for $\mathrm{Re}(\lambda) > \lambda_c$, a problem-dependent critical value. Also, $c(u, \lambda) \sim \exp(\lambda u)$ as $\lambda \to +\infty$, since the largest that x can be in equation (3) is $x = u$. Therefore, the integrand in equation (15) is

$$\exp(-\lambda z_2) \, \left[c(z_1, \lambda)\right]^{N-1} \sim \exp\left[-\lambda\left(z_2 - (N-1)\, z_1\right)\right] \text{ as } \lambda \to +\infty \, . \qquad (16)$$

If contour C is moved far to the right in the λ-plane, where the integrand is guaranteed to be analytic, then the integrand tends to zero if $z_2 > (N-1)\, z_1$. Therefore, SO PDF $p_{\mathbf{z}}(z_1, z_2)$ is zero in that region of field space. The joint PDF in equation (15) can be written as

$$p_{\mathbf{z}}(z_1, z_2) = \left(N \, p(z_1) \, [c(z_1)]^{N-1}\right) \frac{1}{i2\pi} \int_C d\lambda \, \exp(-\lambda z_2) \, \left(\frac{c(z_1, \lambda)}{c(z_1)}\right)^{N-1} , \qquad (17)$$

where

$$c(z_1) = c(z_1,0) = \int_{-\infty}^{z_1} dx\ p(x) \tag{18}$$

is the common FO CDF of RVs $\{\mathbf{x}_n\}$, n=1:N. However, the leading factor in equation (17) is just the FO PDF of the largest RV $\mathbf{z}_1 = \mathbf{x}_1'$. Therefore, the remaining term must be the conditional FO PDF of RV $\mathbf{z}_2$ at argument z_2, given that $\mathbf{z}_1 = z_1$. Expressing

$$\frac{c(z_1,\lambda)}{c(z_1)} = \int_{-\infty}^{z_1} dx\ \frac{p(x)}{c(z_1)} \exp(\lambda x) = \int_{-\infty}^{\infty} dx\ p_c(x) \exp(\lambda x) \ , \tag{19}$$

the quantity $p_c(x)$ is recognized as the conditional PDF of each of the <u>remaining</u> $\{\mathbf{x}_n\}$ RVs, given that the largest one, $\mathbf{x}_1'$, has taken on the value z_1. That is,

$$p_c(x) = \left\{\begin{array}{ll} p(x)/c(z_1) & \text{for } x < z_1 \\ 0 & \text{for } x > z_1 \end{array}\right\} = p_{\mathbf{x}_n}\left(x \mid \mathbf{x}_1' = z_1\right) \ . \tag{20}$$

The corresponding conditional MGF is given in equation (19). Since all the remaining $\{\mathbf{x}_n\}$ RVs are still independent of each other (even given that the largest RV $\mathbf{x}_1' = z_1$), the last term in the integral in equation (17) is the FO MGF of the sum $\mathbf{z}_2$ of the remaining N-1 RVs, given that RV $\mathbf{z}_1$ equals value z_1. Finally, the inverse Laplace transform in equation (17) yields the conditional FO PDF (at argument z_2) of RV $\mathbf{z}_2$, which is the sum of the remaining RVs, given that RV $\mathbf{z}_1$ has taken on the value z_1.

JOINT PDF OF THE m-th LARGEST RV AND SUM OF REMAINING RVs

Again, M = 2, while the RVs of interest are now

$$\mathbf{z}_1 = x'_m \ , \qquad \mathbf{z}_2 = \sum_{n=1}^{N}{}' \ x'_n \ , \tag{21}$$

where the tic mark on the sum denotes $n \neq m$. The SO MGF of RV $\mathbf{z} = [\mathbf{z}_1 \ \mathbf{z}_2]$ is

$$\mu_{\mathbf{z}}(\lambda_1,\lambda_2) = E \exp\left(\lambda_1 x'_m + \lambda_2 \sum_n{}' \ x'_n\right)$$

$$= \int\cdots\int du_1\cdots du_N \ g(u_1,\ldots,u_N) \exp\left(\lambda_1 u_m + \lambda_2 \sum_n{}' \ u_n\right)$$

$$= N! \int_{-\infty}^{\infty} du_1 \ p(u_1) \exp(\lambda_2 u_1) \int_{-\infty}^{u_1} du_2 \ p(u_2) \exp(\lambda_2 u_2) \cdots$$

$$\times \int_{-\infty}^{u_{m-2}} du_{m-1} \ p(u_{m-1}) \exp(\lambda_2 u_{m-1}) \int_{-\infty}^{u_{m-1}} du_m \ p(u_m) \exp(\lambda_1 u_m)$$

$$\times \int_{-\infty}^{u_m} du_{m+1} \ p(u_{m+1}) \exp(\lambda_2 u_{m+1}) \cdots \int_{-\infty}^{u_{N-1}} du_N \ p(u_N) \exp(\lambda_2 u_N) \ . \tag{22}$$

The N-m innermost integrals immediately integrate to

$$\frac{1}{(N-m)!} \left[c(u_m,\lambda_2)\right]^{N-m} \ . \tag{23}$$

The remaining m integrals in equation (22) require that $u_1 > u_2 > \cdots > u_m$. Expressing this requirement (see appendix B) as $u_m < u_{m-1} < \cdots < u_2 < u_1$ enables the alternative form for the

SO MGF to be written as

$$\mu_{\mathbf{z}}(\lambda_1,\lambda_2) = \frac{N!}{(N-m)!}\int_{-\infty}^{\infty} du_m \; p(u_m) \; \exp(\lambda_1 u_m) \; [c(u_m,\lambda_2)]^{N-m}$$

$$\times \int_{u_m}^{\infty} du_{m-1} \; p(u_{m-1}) \; \exp(\lambda_2 u_{m-1}) \cdots \int_{u_2}^{\infty} du_1 \; p(u_1) \; \exp(\lambda_2 u_1) \; . \qquad (24)$$

The m-1 innermost integrals are simply

$$\frac{1}{(m-1)!} \; [e(u_m,\lambda_2)]^{m-1} \; . \qquad (25)$$

The SO MGF of RV $\mathbf{z}$ then follows as

$$\mu_{\mathbf{z}}(\lambda_1,\lambda_2) = m\binom{N}{m}\int_{-\infty}^{\infty} du \; p(u) \; \exp(\lambda_1 u) \; [c(u,\lambda_2)]^{N-m} \; [e(u,\lambda_2]^{m-1} . \qquad (26)$$

For $m = 1$, this result reduces to the earlier special case in equation (13).

The SO PDF of RV $\mathbf{z}$ is given by the two-dimensional inverse Laplace transform of equation (26), namely,

$$p_{\mathbf{z}}(z_1,z_2) = \frac{1}{(i2\pi)^2}\iint_C d\lambda_1 \; d\lambda_2 \; \exp(-\lambda_1 z_1 - \lambda_2 z_2) \; \mu_{\mathbf{z}}(\lambda_1,\lambda_2)$$

$$= m\binom{N}{m} \; p(z_1) \; \frac{1}{i2\pi}\int_C d\lambda \; \exp(-\lambda z_2) \; [c(z_1,\lambda)]^{N-m} \; [e(z_1,\lambda)]^{m-1} \; , \qquad (27)$$

where equation (14) was used. For $m = 1$, this result reduces to that in equation (15). For $m \geq 2$, there is no limit on the

ranges of components z_1, z_2 of field point z (except where PDF value $p(z_1) = 0$). That is, RV $\mathbf{z}_2$ in equation (21) can be arbitrarily larger than RV $\mathbf{z}_1$, and the corresponding PDF value $p_{\mathbf{z}}(z_1, z_2)$ can be nonzero. In terms of the integral in equation (27) for $p_{\mathbf{z}}(z_1, z_2)$, the difference for $m \geq 2$ is that $e(z_1, \lambda)$ is only analytic for $\mathrm{Re}(\lambda) < \lambda_e$, a problem-dependent critical value. Any attempt to move contour C to the right in the λ-plane will encounter a singularity of the analytic continuation of $e(z_1, \lambda)$ at $\mathrm{Re}(\lambda) = \lambda_e$, thereby leading to a nonzero integral result for $p_{\mathbf{z}}(z_1, z_2)$ at any z_1, z_2.

The SO PDF of RV $\mathbf{z}$ can be written as

$$p_{\mathbf{z}}(z_1, z_2) = m \binom{N}{m} p(z_1) \left[c(z_1)\right]^{N-m} \left[e(z_1)\right]^{m-1}$$

$$\times \frac{1}{i2\pi} \int_C d\lambda \exp(-\lambda z_2) \left(\frac{c(z_1, \lambda)}{c(z_1)}\right)^{N-m} \left(\frac{e(z_1, \lambda)}{e(z_1)}\right)^{m-1} . \tag{28}$$

The first factor (upper line) is the FO PDF of the m-th largest RV $\mathbf{z}_1 = \mathbf{x}'_m$. That is, any one of the N RVs $\{\mathbf{x}_n\}$ could take on value z_1; then, there are $(N-1|m-1)$ possibilities for which m-1 RVs lie above z_1 and the remaining N-m RVs lie below z_1. Thus, the total number is $N\ (N-1|m-1) = m\ (N|m)$, with each of these possibilities having probability $[c(z_1)]^{N-m}\ [e(z_1)]^{m-1}$.

The second factor in equation (28) is the conditional PDF of RV $\mathbf{z}_2$ at argument z_2, given that RV $\mathbf{z}_1 = z_1$. It is also the

inverse Laplace transform of the conditional FO MGF of the sum $\mathbf{z}_2$ of the remaining N-1 RVs, given that $\mathbf{x}'_m = \mathbf{z}_1 = z_1$. The ratio

$$\frac{e(z_1,\lambda)}{e(z_1)} = \int_{z_1}^{\infty} dx\, \frac{p(x)}{e(z_1)} \exp(\lambda x) = \int_{-\infty}^{\infty} dx\, p_e(x) \exp(\lambda x) \tag{29}$$

is the conditional MGF of each of the remaining RVs that lie above z_1, and

$$p_e(x) = \left\{ \begin{array}{rl} p(x)/e(z_1) & \text{for } x > z_1 \\ 0 & \text{for } x < z_1 \end{array} \right\} \tag{30}$$

is the corresponding conditional PDF, given that $\mathbf{x}'_m = \mathbf{z}_1 = z_1$. Since all of the m-1 remaining RVs that lie above value z_1 are independent of each other, and all of the N-m remaining RVs that lie below value z_1 are independent of each other, the conditional MGFs can be multiplied together to give the conditional MGF of the remaining sum RV. Finally, the inverse Laplace transform in equation (28) gives the conditional PDF of interest.

JOINT PDF OF THE TWO LARGEST RVs AND SUM OF REMAINING RVs

For this special case, M = 3 and the RVs of interest are

$$\mathbf{z}_1 = \mathbf{x}_1' , \quad \mathbf{z}_2 = \mathbf{x}_2' , \quad \mathbf{z}_3 = \sum_{n=3}^{N} \mathbf{x}_n' . \tag{31}$$

The joint third-order (TO) MGF of RV $\mathbf{z} = [\mathbf{z}_1 \; \mathbf{z}_2 \; \mathbf{z}_3]$ is

$$\mu_z(\lambda_1,\lambda_2,\lambda_3) = \mathrm{E} \exp\left(\lambda_1 \mathbf{x}_1' + \lambda_2 \mathbf{x}_2' + \lambda_3 \sum_{n=3}^{N} \mathbf{x}_n'\right)$$

$$= \int\cdots\int du_1 \cdots du_N \; g(u_1,\ldots,u_N) \exp\left(\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 \sum_{n=3}^{N} u_n\right)$$

$$= N! \int_{-\infty}^{\infty} du_1 \; p(u_1) \exp(\lambda_1 u_1) \int_{-\infty}^{u_1} du_2 \; p(u_2) \exp(\lambda_2 u_2)$$

$$\times \int_{-\infty}^{u_2} du_3 \; p(u_3) \exp(\lambda_3 u_3) \cdots \int_{-\infty}^{u_{N-1}} du_N \; p(u_N) \exp(\lambda_3 u_N)$$

$$= N(N-1) \int_{-\infty}^{\infty} du_1 \; p(u_1) \exp(\lambda_1 u_1) \int_{-\infty}^{u_1} du_2 \; p(u_2) \exp(\lambda_2 u_2) \; c(u_2,\lambda_3)^{N-2} . \tag{32}$$

The corresponding TO PDF is given by the three-dimensional inverse Laplace transform

$$p_z(z_1,z_2,z_3) = \frac{1}{(i2\pi)^3} \iiint_C d\lambda_1 \; d\lambda_2 \; d\lambda_3 \exp(-\lambda_1 z_1 - \lambda_2 z_2 - \lambda_3 z_3)$$

$$\times \mu_z(\lambda_1,\lambda_2,\lambda_3) . \tag{33}$$

Substitution of μ_z from equation (32) into equation (33), an

interchange of integrals, and use of equation (14) results in

$$p_{\mathbf{z}}(z_1,z_2,z_3) = \frac{N(N-1)}{i2\pi} \int_C d\lambda_3 \exp(-\lambda_3 z_3)$$

$$\times \int_{-\infty}^{\infty} du_1 \; p(u_1) \; \delta(u_1-z_1) \int_{-\infty}^{u_1} du_2 \; p(u_2) \; [c(u_2,\lambda_3)]^{N-2} \; \delta(u_2-z_2) \; . \quad (34)$$

The innermost integral on u_2 is zero if $z_2 > u_1$. Therefore, its value is $p(z_2) \; [c(z_2,\lambda_3)]^{N-2} \; U(u_1-z_2)$, which gives the TO PDF

$$p_{\mathbf{z}}(z_1,z_2,z_3) = N(N-1) \; p(z_1) \; p(z_2) \; U(z_1-z_2)$$

$$\times \frac{1}{i2\pi} \int_C d\lambda \exp(-\lambda z_3) \; [c(z_2,\lambda)]^{N-2} \; . \quad (35)$$

The unit-step function U( ) merely emphasizes that field point $z = [z_1 \; z_2 \; z_3]$ should have components $z_1 > z_2$ because RVs $\mathbf{z}_1 > \mathbf{z}_2$ always; that is, $\mathbf{x}_1' > \mathbf{x}_2'$, by definition. The joint PDF must be zero if field point components $z_1 < z_2$.

Initially, there is no obvious limit on component z_3 of field point z; however, because RVs $\mathbf{x}_n' < \mathbf{x}_2' = \mathbf{z}_2$ for n=3:N, then

$$\sum_{n=3}^{N} \mathbf{x}_n' < (N-2) \; \mathbf{z}_2 \; . \quad (36)$$

Therefore, PDF $p_{\mathbf{z}}(z_1,z_2,z_3)$ will be nonzero for field point components $z_3 < (N-2) \; z_2$; any choice for field point component z_3 larger than $(N-2) \; z_2$ will result in a zero value for PDF $p_{\mathbf{z}}$.

This same result may be deduced from the integral relation for PDF p_z in equation (35); the analysis is identical to that in the discussion surrounding equation (16). In summary, field point z in PDF $p_z(z)$ should satisfy $z_1 > z_2 > z_3/(N-2)$ to realize nonzero values for $p_z(z)$.

Equation (35) can be written in the form

$$p_z(z_1,z_2,z_3) = N(N-1)\ p(z_1)\ p(z_2)\ [c(z_2)]^{N-2}$$

$$\times \frac{1}{i2\pi} \int_C d\lambda\ \exp(-\lambda z_3) \left(\frac{c(z_2,\lambda)}{c(z_2)}\right)^{N-2} . \tag{37}$$

The leading factor (upper line) is the joint SO PDF of $\mathbf{x}_1' = \mathbf{z}_1$ and $\mathbf{x}_2' = \mathbf{z}_2$ at arguments $z_1 > z_2$. The last factor is the conditional PDF of RV $\mathbf{z}_3$ at argument z_3, given values for RVs $\mathbf{z}_1$ and $\mathbf{z}_2$. This reasoning is identical to that presented earlier.

JOINT PDF OF n_1-th LARGEST RV, n_2-th LARGEST RV,..., n_{M-1}-th LARGEST RV, AND SUM OF REMAINING RVs

Without loss of generality, $1 \le n_1 < n_2 < \cdots < n_{M-1} \le N$ and $M \ge 2$. The first step is to identify the RVs

$$\mathbf{z}_1 = \mathbf{x}'_{n_1}, \quad \ldots, \quad \mathbf{z}_{M-1} = \mathbf{x}'_{n_{M-1}}, \quad \mathbf{z}_M = \sum_{n=1}^{N}{}' \mathbf{x}'_n , \tag{38}$$

where the tic mark denotes $n \ne n_1, n_2, \ldots, n_{M-1}$. Then, the field point $z = [z_1 \; z_2 \; \cdots \; z_{M-1} \; z_M]$ is taken such that $z_1 \ge z_2 \ge \cdots \ge z_{M-1}$.

These events can only occur if the following conditions are met: n_1-1 RVs must lie above z_1; n_2-n_1-1 RVs must lie between z_2 and z_1; n_3-n_2-1 RVs must lie between z_3 and z_2;...; $n_{M-1}-n_{M-2}-1$ RVs must lie between z_{M-1} and z_{M-2}; and $N-n_{M-1}$ RVs must lie below z_{M-1}. The number of ways in which these events can happen is

$$F = \frac{N!}{(n_1-1)! \; (n_2-n_1-1)! \; \cdots \; (n_{M-1}-n_{M-2}-1)! \; (N-n_{M-1})!} , \tag{39}$$

where $M \ge 2$.

If an "interval" MGF is defined as

$$\mu(u_a,u_b,\lambda) = \int_{u_a}^{u_b} dx \; p(x) \; \exp(\lambda x) = c(u_b,\lambda) - c(u_a,\lambda)$$

$$= e(u_a,\lambda) - e(u_b,\lambda) , \tag{40}$$

then the joint PDF of RV $\mathbf{z} = [\mathbf{z}_1 \; \cdots \; \mathbf{z}_M]$ at field point z is

$$p_z(z_1,\ldots,z_M) = F\ p(z_1)\ \cdots\ p(z_{M-1})\ \frac{1}{i2\pi}\int_C d\lambda\ \exp(-\lambda z_M)$$

$$\times\ [c(z_{M-1},\lambda)]^{N-n_{M-1}}\ [\mu(z_{M-1},z_{M-2},\lambda)]^{n_{M-1}-n_{M-2}-1} \times \cdots$$

$$\times\ [\mu(z_2,z_1,\lambda)]^{n_2-n_1-1}\ [e(z_1,\lambda)]^{n_1-1}\ . \tag{41}$$

This result has been deduced as the natural extension of the earlier multiple integral approaches and their special cases in the previous section.

An alternative form, which is more amenable to programming, is

$$p_z(z_1,\ldots,z_M) = \frac{N!}{D}\prod_{m=1}^{M-1}\{p(z_m)\}\ \frac{1}{i2\pi}\int_C d\lambda\ \exp(-\lambda z_M)\ I(\lambda)\ , \tag{42}$$

where constant

$$D = (n_1-1)!\ \prod_{m=1}^{M-2}\{(n_{m+1}-n_m-1)!\}\ (N-n_{M-1})! \tag{43}$$

and

$$I(\lambda) = [e(z_1,\lambda)]^{n_1-1}\ \prod_{m=1}^{M-2}\left\{[e(z_{m+1},\lambda) - e(z_m,\lambda)]^{n_{m+1}-n_m-1}\right\}$$

$$\times\ [c(z_{M-1},\lambda)]^{N-n_{M-1}}\ . \tag{44}$$

For each λ, it is necessary to compute $\{e(z_m,\lambda)\}$ for $m=1:M-1$ and $c(z_{M-1},\lambda) = \mu(\lambda) - e(z_{M-1},\lambda)$.

Letting $\lambda = \lambda_s + iy$, where λ_s is (fairly near) the real saddlepoint location of the integrand of equation (42), then the joint PDF in equation (42) can be expressed as

$$p_z(z_1,\ldots,z_M) = \frac{N!}{D\pi} \prod_{m=1}^{M-1} \{p(z_m)\} \int_0^\infty dy \ \mathrm{Re}\{\exp[-(\lambda_s+iy)z_M] \ I(\lambda_s+iy)\} \ . \tag{45}$$

COMMON NONLINEAR TRANSFORMATION OF REMAINING RVs

If all the remaining N+1−M RVs in the last term of equation (38) are subjected to a common nonlinear transformation, $h(x)$, prior to summation, the pertinent $e(u,\lambda)$ and $c(u,\lambda)$ functions above must be replaced by

$$e_h(u,\lambda) = \int_u^\infty dx \ p(x) \ \exp[\lambda \ h(x)] \ ,$$

$$c_h(u,\lambda) = \int_{-\infty}^u dx \ p(x) \ \exp[\lambda \ h(x)] = e_h(-\infty,\lambda) - e_h(u,\lambda) \ . \tag{46}$$

For example, if $x_n = |g_n|$, where $g_n = N(0,1)$, and if $h(x) = x^2$, then for $\mathrm{Re}(\lambda) < 1/2$,

$$e_h(u,\lambda) = \int_u^\infty dx \ \frac{2}{(2\pi)^{1/2}} \exp(-x^2/2) \ \exp(\lambda x^2)$$

$$= \frac{2}{(1-2\lambda)^{1/2}} \ \Phi\left(-u \ (1-2\lambda)^{1/2}\right) \ \text{for } u > 0; \quad \frac{1}{(1-2\lambda)^{1/2}} \ \text{for } u < 0 \ . \tag{47}$$

Also,

$$c_h(u,\lambda) = \frac{1 - 2\Phi\left(-u\ (1-2\lambda)^{\frac{1}{2}}\right)}{(1-2\lambda)^{\frac{1}{2}}} \text{ for } u > 0; \quad 0 \text{ for } u < 0 \ . \tag{48}$$

These are not the same functions encountered in example 4 in appendix A; rather, the functions here can be expressed as

$$e_h(u,\lambda) = e_4(u^2,\lambda) \ , \quad c_h(u,\lambda) = c_4(u^2,\lambda) \ . \tag{49}$$

In general, letting RV $\mathbf{y} = h(\mathbf{x})$, there follows from equation (46)

$$e_h(u,\lambda) = \int_u^\infty dx\ p_{\mathbf{x}}(x)\ \exp[\lambda\ h(x)] = \int_{h(u)}^\infty dy\ p_{\mathbf{y}}(y)\ \exp(\lambda y)$$

$$\equiv e_{\mathbf{y}}(h(u),\lambda) \ . \tag{50}$$

RATIOS OF RANDOM VARIABLES

The ratios of RVs are defined according to

$$\mathbf{r}_m = \mathbf{z}_m/\mathbf{z}_M \quad \text{for } m=1:M-1 \ . \tag{51}$$

Then the joint PDF of $[\mathbf{r}_1 \cdots \mathbf{r}_{M-1}\ \mathbf{z}_M]$ is

$$\tilde{p}(r_1,\ldots,r_{M-1},z_M) = (z_M)^{M-1}\ p_{\mathbf{z}}(z_M r_1,\ldots,z_M r_{M-1},z_M) \ , \tag{52}$$

while the PDF of RV $\mathbf{r} = [\mathbf{r}_1 \cdots \mathbf{r}_{M-1}]$ is

$$p_{\mathbf{r}}(r_1,\ldots,r_{M-1}) = \int_{-\infty}^{\infty} du\ u^{M-1}\ p_{\mathbf{z}}(ur_1,\ldots,ur_{M-1},u) \ . \tag{53}$$

It is necessary to carry out this one-dimensional integral on u to evaluate the PDF at the M-1 dimensional field point $r = [r_1 \cdots r_{M-1}]$ of interest.

SUMMARY

The solutions for the joint PDFs of several simpler ordered statistics problems were obtained by first deriving the joint MGFs of the RVs of interest. Then, the multidimensional inverse Laplace transform back into the PDF domain was manipulated into a form that involved only a single contour integral in the MGF domain. Interpretation of the final form in terms of conditional MGFs and conditional PDFs afforded useful insights on the meanings of the various quantities involved, as well as on their interactions with each other. Finally, these results were extended to the general problem of M-1 ordered statistics and the sum of the remaining RVs in an original set of N IID RVs with arbitrary common PDF p(x).

Numerical evaluation of the joint PDF is most easily accomplished by locating (approximately) the real saddlepoint of the MGF integrand in the λ-plane, and using a Bromwich contour that passes through this point. It is not necessary to resort to a saddlepoint approximation in the one λ dimension; instead, very high accuracy in the numerical evaluation of the single integral for the joint PDF can be achieved with little computer effort.

An alternative approach to this problem is afforded by determining the joint MGF of the RV $\mathbf{z}$ and then resorting to a saddlepoint approximation to obtain the joint PDF at field point z of interest. This technique is presented in appendix C.

APPENDIX A — EXAMPLES OF $c(u,\lambda)$ AND $e(u,\lambda)$

Equations (3) through (6) of the main text lead to

$$c(u,\lambda) = \int_{-\infty}^{u} dx\ p(x)\ \exp(\lambda x) = \mu(\lambda) - e(u,\lambda)\ ,$$

$$e(u,\lambda) = \int_{u}^{+\infty} dx\ p(x)\ \exp(\lambda x) = \mu(\lambda) - c(u,\lambda)\ ,$$

where $\mu(\lambda)$ is the FO MGF corresponding to FO PDF $p(x)$.

EXAMPLE 1 — EXPONENTIAL

$$p(x) = \exp(-x) \quad \text{for } x > 0,\ 0 \text{ for } x < 0\ ;$$

$$c(u,\lambda) = \frac{1 - \exp[-u(1-\lambda)]}{1-\lambda} \quad \text{for } u > 0,\ 0 \text{ for } u < 0\ ;$$

$$e(u,\lambda) = \frac{\exp[-u(1-\lambda)]}{1-\lambda} \quad \text{for } u > 0,\ \frac{1}{1-\lambda} \quad \text{for } u < 0\ .$$

In the relation for $e(u,\lambda)$, $\mathrm{Re}(\lambda) < 1$ is also required.

EXAMPLE 2 — GAUSSIAN, $\mathbf{x} = \mathbf{g} = N(0,1)$

$$p(x) = (2\pi)^{-\frac{1}{2}} \exp(-x^2/2) \quad \text{for all } x\ ;$$

$$c(u,\lambda) = \exp(\lambda^2/2)\ \Phi(u - \lambda) \quad \text{for all } u,\lambda\ ;$$

$$e(u,\lambda) = \exp(\lambda^2/2)\ \Phi(\lambda - u) \quad \text{for all } u,\lambda\ .$$

The function Φ is the CDF of a normalized Gaussian RV, namely,

$$\Phi(x) = \int_{-\infty}^{x} dt\ (2\pi)^{-\frac{1}{2}} \exp(-t^2/2) \quad \text{for all } x\ .$$

EXAMPLE 3 – MAGNITUDE GAUSSIAN, $\mathbf{x} = |\mathbf{g}|$

$$p(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \exp(-x^2/2) \quad \text{for } x > 0 \;, \quad 0 \text{ for } x < 0 \;;$$

$$c(u,\lambda) = 2 \exp(\lambda^2/2) \; [\Phi(\lambda) - \Phi(\lambda - u)] \quad \text{for } u > 0, \quad 0 \text{ for } u < 0 \;;$$

$$e(u,\lambda) = 2 \exp(\lambda^2/2) \; \Phi(\lambda - u) \text{ for } u>0, \; 2 \exp(\lambda^2/2) \; \Phi(\lambda) \text{ for } u<0 \;.$$

EXAMPLE 4 – SQUARED GAUSSIAN, $\mathbf{x} = \mathbf{g}^2$

$$p(x) = \frac{\exp(-x/2)}{(2\pi x)^{\frac{1}{2}}} \quad \text{for } x > 0, \quad 0 \text{ for } x < 0 \;;$$

$$c(u,\lambda) = \frac{1 - 2\Phi\left(- u^{\frac{1}{2}}(1 - 2\lambda)^{\frac{1}{2}}\right)}{(1 - 2\lambda)^{\frac{1}{2}}} \quad \text{for } u > 0, \quad 0 \text{ for } u < 0 \;;$$

$$e(u,\lambda) = \frac{2\Phi\left(- u^{\frac{1}{2}}(1 - 2\lambda)^{\frac{1}{2}}\right)}{(1 - 2\lambda)^{\frac{1}{2}}} \quad \text{for } u > 0, \quad (1 - 2\lambda)^{-\frac{1}{2}} \text{ for } u < 0.$$

In the relation for $e(u,\lambda)$, $\mathrm{Re}(\lambda) < 0.5$ is also required.

EXAMPLE 5 – CHI-SQUARED RV OF 2k+2 DEGREES OF FREEDOM; k=0,1,2...

$$p(x) = \frac{x^k \exp(-x)}{k!} \quad \text{for } x > 0, \quad 0 \text{ for } x < 0 \;;$$

$$c(u,\lambda) = (1-\lambda)^{-k-1} \left[1 - \exp\left(-u(1-\lambda)\right) \sum_{j=0}^{k} \frac{u^j \; (1-\lambda)^j}{j!}\right] \quad \text{for } u > 0,$$

$$= 0 \text{ for } u < 0 \;;$$

$$e(u,\lambda) = (1-\lambda)^{-k-1} \exp\left(-u(1-\lambda)\right) \sum_{j=0}^{k} \frac{u^j \; (1-\lambda)^j}{j!} \quad \text{for } u > 0,$$

$$= (1-\lambda)^{-k-1} \quad \text{for } u < 0 \;.$$

In the relation for $e(u,\lambda)$, $\mathrm{Re}(\lambda) < 1$ is also required.

For complex arguments of Φ in examples 2 through 4, the relation

$$\Phi(z) = \frac{1}{2} \exp(-z^2/2)\ w(-iz/\surd 2)$$

in terms of the w function (reference 1, chapter 7) is very useful.

The above results extend immediately to a scaled and shifted RV $\tilde{x} = a + b\,x$ according to

$$\tilde{p}(x) = \frac{1}{b}\, p\left(\frac{x - a}{b}\right) ,$$

$$\tilde{c}(u,\lambda) = \exp(a\lambda)\ c\left(\frac{u - a}{b}, b\lambda\right) ,$$

$$\tilde{e}(u,\lambda) = \exp(a\lambda)\ e\left(\frac{u - a}{b}, b\lambda\right) .$$

APPENDIX B — INTERCHANGE OF MULTIPLE INTEGRALS

Suppose a multiple integral over the range $u_a < u_1 < u_2 < u_3 < u_4 < u_b$ is required, namely,

$$I = \int_{u_a}^{u_b} du_1 \int_{u_1}^{u_b} du_2 \int_{u_2}^{u_b} du_3 \int_{u_3}^{u_b} du_4 \; f(u_1,u_2,u_3,u_4) \; . \tag{B-1}$$

This multiple integral may be expressed in numerous equivalent representations, such as

$$I = \int_{u_a}^{u_b} du_4 \int_{u_a}^{u_4} du_3 \int_{u_a}^{u_3} du_2 \int_{u_a}^{u_2} du_1 \; f(u_1,u_2,u_3,u_4)$$

$$= \int_{u_a}^{u_b} du_3 \int_{u_a}^{u_3} du_2 \int_{u_a}^{u_2} du_1 \int_{u_3}^{u_b} du_4 \; f(u_1,u_2,u_3,u_4)$$

$$= \int_{u_a}^{u_b} du_2 \int_{u_2}^{u_b} du_4 \int_{u_a}^{u_2} du_1 \int_{u_2}^{u_4} du_3 \; f(u_1,u_2,u_3,u_4)$$

$$= \int_{u_a}^{u_b} du_1 \int_{u_1}^{u_b} du_4 \int_{u_1}^{u_4} du_2 \int_{u_2}^{u_4} du_3 \; f(u_1,u_2,u_3,u_4) \; . \tag{B-2}$$

Each pair of limits is chosen to be as tight as possible. The general rule is that if a potential limit variable has already been integrated out, the next tightest limit variable still active should be selected.

APPENDIX C — JOINT MGF OF SELECTED ORDERED SET AND REMAINDER

The RVs of interest are those listed in equation (38) and its sequel, except that the original RVs $\{x_n\}$ are specialized here to be exponential RVs with common PDF exp(-u) for u > 0. The joint MGF of the ordered RVs $\{x'_n\}$, n=1:N, is shown in reference 2 (page 68, equation (B-18)) to be

$$\mu_o(\alpha_1,\ldots,\alpha_N) = \frac{N!}{(1-\alpha_1)(2-\alpha_1-\alpha_2)\cdots(N-\alpha_1-\cdots-\alpha_N)} \tag{C-1}$$

for $\alpha_1 < 1$, $\alpha_1 + \alpha_2 < 2,\ldots,$ $\alpha_1 + \cdots + \alpha_N < N$. Alternatively, use of the N×1 vector $\alpha = [\alpha_1 \cdots \alpha_N]^T$ produces

$$\mu_o(\alpha) = 1\Big/\prod_{n=1}^{N}\phi_n(\alpha_1,\ldots,\alpha_N) \;, \tag{C-2}$$

where

$$\phi_n(\alpha) = 1 - \sum_{p=1}^{N} q_{np}\,\alpha_p \quad \text{for } n=1:N \tag{C-3}$$

and

$$q_{np} = \begin{Bmatrix} 1/n & \text{for } p=1:n \\ 0 & \text{for } p=n+1:N \end{Bmatrix} \quad \text{for } n=1:N \;. \tag{C-4}$$

Now, the N×N matrix $Q = [q_{np}]$ is defined. Then, N×1 vector

$$\phi(\alpha) = \begin{bmatrix} \phi_1(\alpha) \\ \vdots \\ \phi_N(\alpha) \end{bmatrix} = \underline{1} - Q\,\alpha \;, \tag{C-5}$$

where $\underline{1}$ is an N×1 vector of ones. All the components $\{\phi_n(\alpha)\}$ must be positive.

The joint MGF $\mu(\lambda)$ of the RVs of interest in equation (38) is given by $\mu_o(\alpha_1,\ldots,\alpha_N)$, with every α_n replaced by λ_M (that is, $\alpha_n \to \lambda_M$), except for

$$\alpha_{n_1} \to \lambda_1 \ , \quad \alpha_{n_2} \to \lambda_2 \ ,\ldots, \ \alpha_{n_{M-1}} \to \lambda_{M-1} \ . \qquad \text{(C-6)}$$

With M×1 vector $\lambda = [\lambda_1 \ \cdots \ \lambda_M]^T$, this replacement corresponds to the transformation $\alpha = A\ \lambda$, where N×M matrix A has ones in the M-th column, except for zeros in rows n_1, $n_2,\ldots,$ n_{M-1}, and A has ones in row n_1, column 1; row n_2, column 2;...; and row n_{M-1}, column M-1. The result is joint MGF $\mu(\lambda) = 1/\Pi\phi_n(\alpha = A\ \lambda)$, where

$$\phi(\alpha) = \phi(A\ \lambda) = \underline{1} - P\ \lambda \ , \qquad P = Q\ A = [p_{nm}] \ . \qquad \text{(C-7)}$$

The matrix P is N×M.

The joint cumulant generating function (CGF) corresponding to joint MGF $\mu(\lambda)$ is

$$\chi(\lambda) = -\sum_{n=1}^{N} \log\Big(\phi_n(A\ \lambda)\Big) \ , \qquad \text{(C-8)}$$

where

$$\phi_n(A\ \lambda) = 1 - \sum_{m=1}^{M} p_{nm}\ \lambda_m \quad \text{for } n=1:N \ . \qquad \text{(C-9)}$$

All N of these quantities $\{\phi_n\}$ must be positive.

For purposes of evaluating the saddlepoint approximation and a correction term to the joint PDF corresponding to the joint MGF $\mu(\lambda)$, the following partial derivatives of the joint CGF are

needed. Letting $\phi_n = \phi_n(A\ \lambda)$ and $b_{nm} = p_{nm}/\phi_n$ for n=1:N, m=1:M, then

$$\frac{\partial \phi_n(A\ \lambda)}{\partial \lambda_m} = -\ p_{nm} \quad \text{for } n=1:N,\ m=1:M\ ,$$

$$\frac{\partial}{\partial \lambda_m}\ \chi(\lambda) = \sum_{n=1}^{N} b_{nm}\ ,$$

$$\frac{\partial^2}{\partial \lambda_m\ \partial \lambda_\ell}\ \chi(\lambda) = \sum_{n=1}^{N} b_{nm}\ b_{n\ell}\ ,$$

$$\frac{\partial^3}{\partial \lambda_m\ \partial \lambda_\ell\ \partial \lambda_k}\ \chi(\lambda) = 2\sum_{n=1}^{N} b_{nm}\ b_{n\ell}\ b_{nk}\ ,$$

$$\frac{\partial^4}{\partial \lambda_m\ \partial \lambda_\ell\ \partial \lambda_k\ \partial \lambda_j}\ \chi(\lambda) = 6\sum_{n=1}^{N} b_{nm}\ b_{n\ell}\ b_{nk}\ b_{nj}\ .$$

REFERENCES

1. **Handbook of Mathematical Functions**, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, No. 55, U.S. Government Printing Office, Washington, DC, June 1964.

2. A. H. Nuttall, "Detection Performance of Generalized Likelihood Ratio Processors for Random Signals of Unknown Location, Structure, Extent, and Strength," NUWC-NPT Technical Report 10,739, Naval Undersea Warfare Center Division, Newport, RI, 25 August 1994.

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